(9) Tensor operators

① Scalar and vector operators : the definition

- Scalar operator : $\mathcal{D}^\dagger(R)\, S\, \mathcal{D}(R) = S \iff [J_i, S] = 0$
- Vector operator : $\mathcal{D}^\dagger(R)\, \vec{V}\, \mathcal{D}(R) = R\vec{V} \iff [J_i, V_j] = i\hbar\varepsilon_{ijk}V_k$

② Tensor operators

- Scalar : rank 0 , Vector : rank 1 .
- Rank-n Cartesian Tensor operator : $T_{\underbrace{\cdots ijk\cdots}_{n\text{-indices}}}$ $(\cdots i, j, k, \cdots = 1, 2, 3)$

Rotation : $$T_{ijk\cdots} \to \sum_{\cdots i', j', k', \cdots} \cdots R_{ii'} R_{jj'} R_{kk'} \cdots T_{\cdots i'j'k'\cdots}$$

→ very complicated ! But, it can be simpler in practice.

ex. a "dyadic" tensor : $T_{ij} = U_i V_j$ (rank 2)

→ can be decomposed into 3 separated rotations.

$$U_iV_j = \underbrace{\frac{\vec{U}\cdot\vec{V}}{3}\delta_{ij}}_{\text{Scalar op.}} + \underbrace{\frac{U_iV_j - U_jV_i}{2}}_{\substack{\text{anti-symmetric} \\ \sim \varepsilon_{ijk}(\vec{U}\times\vec{V})_k}} + \underbrace{\left(\frac{U_iV_j + U_jV_i}{2} - \frac{\vec{U}\cdot\vec{V}}{3}\delta_{ij}\right)}_{\substack{3\times 3 \text{ symmetrical} \\ \text{Traceless tensor.}}}$$

Scalar op.	anti-symmetric	3×3 symmetrical Traceless tensor.
↓	↓	↓
1 variable : Scalar op.	3 var. : vector	5 variables : rank-2.

"irreducible" subspaces ⇓ "Reduced"

$$\boxed{\begin{array}{ccc} \dim = 1 & \dim. = 3 & \dim. = 5 \\ (l = 0) & (l = 1) & (l = 2) \end{array}}$$

→ Number of independent components :

$$\underset{\text{reducible}}{\underline{3\times3}} = \underline{1} + \underline{3} + \underline{5} \quad \text{irreducible.}$$

$$\rightarrow (\ell=1)\otimes(\ell=1) = (\ell=0)\oplus(\ell=1)\oplus(\ell=2)$$

In terms of the irreducible spherical tensors.

③ Vector operator as a spherical tensor.

- Vector operator revisited.

def. $\mathcal{D}^{\dagger}(R)\,\vec{V}\,\mathcal{D}(R) = R\,\vec{V} \longleftrightarrow [J_i, V_j] = i\hbar\,\varepsilon_{ijk} V_k$

Rotation in the Hilbert space

Rotation in the physical space.

$$\rightarrow \mathcal{D}(R) = \exp\left[-\frac{i}{\hbar}\theta(\hat{n}\cdot\vec{J})\right]$$

$$\rightarrow R = \exp\left[-i\theta(\hat{n}\cdot\vec{\mathcal{J}})\right]$$

$\ell=1$, $\langle m|\vec{J}|m'\rangle$

"In Cartesian basis"

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad \mathcal{J}_x = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}$$

⌈See Lecture 18.⌋

$(\mathcal{J}_i)_{jk} = -i\varepsilon_{ijk}$

$$J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} \qquad \mathcal{J}_y = \begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix}$$

$$J_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} \qquad \mathcal{J}_z = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}$$

DIFFERENT !

" Rotation of a vector in 3D

$\Rightarrow \boxed{\ell=1}$ "

!!!

BUT $\boxed{\vec{J} = U^{\dagger}\vec{\mathcal{J}}\,U}$

Spherical basis ↑ ↑ Cartesian basis.

U : unitary transformation

→ $\vec{J}$ corresponds to spin-1 angular momentum in the Cartesian basis.

See, also, $\vec{J}^2 = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}$ → eigenvalues ⇒ $2 = j(j+1)$

→ $j = 1$.

(Any classical vector field $A(\vec{x})$, like a photon corresponds to spin-1.)

• Spherical basis ← the eigenvectors of J_z.

def. $\hat{e}_1 = -\dfrac{\hat{x}+i\hat{y}}{\sqrt{2}}$, $\hat{e}_0 = \hat{z}$, $\hat{e}_{-1} = \dfrac{\hat{x}-i\hat{y}}{\sqrt{2}}$

covariant form.

↕

$(l=1)$ $Y_1^{\pm1} = \mp\sqrt{\dfrac{3}{4\pi}}\dfrac{x \pm iy}{\sqrt{2}\,r}$, $Y_1^0 = \sqrt{\dfrac{3}{4\pi}}\dfrac{z}{r}$

⌜Spherical Harmonics ∈ irreducible spherical tensors⌟
[$Y_l^m(\vec{V})$]

Let's see if $\hat{e}_q$ indeed belongs to $j=1$.

① $J_z \hat{e}_q = q\,\hat{e}_q$ ∥ $q = 0, \pm1$ ⟷ $J_z|l,m\rangle = m\hbar|l,m\rangle$

② $J_\pm \hat{e}_q = \sqrt{(1\mp q)(1\pm q+1)}\,\hat{e}_{q\pm1}$ ∥ $J_\pm = J_x \pm iJ_y$

∴ $\hat{e}_q$ indeed works like $|l=1, m\rangle$ on Y_1^m.

properties of the spherical basis

① $\hat{e}_q = (-1)^q \hat{e}^*_{-q}$

② orthogonality : $\hat{e}^*_{q'} \cdot \hat{e}_q = \delta_{qq'}$

③ Identity $I = \sum_q \hat{e}^*_q \hat{e}_q = \sum_q \hat{e}_q \hat{e}^*_q$

contravar. (tensor product) ↳ covariant

⊕ Vector $\vec{X} = \sum_q \hat{e}_q^* X_q$, $X_q = \hat{e}_q \cdot \vec{X}$

… contravariant.

covariant … $\vec{X} = \sum_q \hat{e}_q X_q$, $X_q = \hat{e}_q^* \cdot \vec{X}$

→ Rotation : $R\hat{e}_q = \sum_{q',q''} \hat{e}_{q'} \underbrace{\hat{e}_{q'}^* R \hat{e}_{q''}}_{} \underbrace{\hat{e}_{q''}^* \cdot \hat{e}_q}_{= \delta_{q''q}}$

$= \hat{e}_{q'}^* e^{-i\vec{\theta}\cdot\vec{J}} \hat{e}_{q''}$

$= \langle 1, q' | e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}} | 1, q'' \rangle$

$= \mathcal{D}^{(1)}_{q'q''}(R)$

$\therefore \boxed{R\hat{e}_q = \sum_{q'} \hat{e}_{q'} \mathcal{D}^{(1)}_{q'q}(R)}$

- Irreducible spherical tensor of order 1.

$V_q \equiv T^{(1)}_q = \hat{e}_q \cdot \vec{V}$

rotation : $\mathcal{D}^\dagger(R) T^{(1)}_q \mathcal{D}(R) = \hat{e}_q \cdot R\vec{V}$

$= (R^{-1}\hat{e}_q) \cdot \vec{V} = \sum_{q'} \hat{e}_{q'} \mathcal{D}^{(1)}_{q'q}(R^{-1}) \cdot \vec{V}$

by setting $R \to R^{-1}$,

$$\mathcal{D}(R)\, T^{(1)}_q\, \mathcal{D}^\dagger(R) = \sum_{q'} T^{(1)}_{q'} \mathcal{D}^{(1)}_{q'q}(R)$$

!

④ Irreducible spherical tensor operator.

generalization

def. $\boxed{\mathcal{D}(R)\, T^{(k)}_q\, \mathcal{D}^\dagger(R) = \sum_{q'=-k}^{k} T^{(k)}_{q'} \mathcal{D}^{(k)}_{q'q}(R)}$

‖ k : rank / order → non-negative INTEGER ☆☆☆

(It's the rotation in the physical space!)

↕

def. Commutation relation

- $[J_z, T^{(k)}_q] = \hbar q\, T^{(k)}_q$
- $[J_\pm, T^{(k)}_q] = \hbar \sqrt{(k \mp q)(k \pm q + 1)}\; T^{(k)}_{q\pm 1}$

proof with infinitesimal rotations $\mathcal{D}(R) \simeq 1 - \frac{i}{\hbar}\theta(\vec{J}\cdot\hat{n})$

$$\left(1 - \frac{i}{\hbar}\theta(\vec{J}\cdot\hat{n})\right) T_q^{(k)} \left(1 + \frac{i}{\hbar}\theta(\vec{J}\cdot\hat{n})\right)$$
$$= \sum_{q'=-k}^{k} T_{q'}^{(k)} \langle k, q' | \left(1 - \frac{i}{\hbar}\theta(\vec{J}\cdot\hat{n})\right) | k, q \rangle$$

$$\Rightarrow \left[\vec{J}\cdot\hat{n}, T_q^{(k)}\right] = \sum_{q'} T_{q'}^{(k)} \langle k, q' | \vec{J}\cdot\hat{n} | k, q \rangle$$ //

Choose $\hat{n} = \hat{z}$ to get $[J_z, T_q^{(k)}]$; do similarly for $J_\pm$.

also, one can prove another commutation relation:

$$\Rightarrow \sum_i [J_i, [J_i, T_q^{(k)}]] = \hbar^2 k(k+1) T_q^{(k)}.$$

⑤ Product of the irreducible spherical tensors

$$T_q^{(k)} = \sum_{q_1, q_2} \underbrace{\langle k_1 k_2 ; q_1 q_2 | k_1 k_2 ; k\, q \rangle}_{= \text{C-G coeff.}} X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)}$$

It's just like the addition of angular momenta ...

Ex. $T_0^{(0)} = -\frac{1}{3}\vec{U}\cdot\vec{V}$ $\qquad U_q, V_q$: rank-1 spherical tensors.

$T_q^{(1)} = \frac{1}{i\sqrt{2}}(\vec{U}\times\vec{V})_q$

$T_{\pm 2}^{(2)} = U_{\pm 1} V_{\pm 1}$

$T_{\pm 1}^{(2)} = \frac{1}{\sqrt{2}}(U_{\pm 1}V_0 + U_0 V_{\pm 1})$

$T_0^{(2)} = \frac{1}{\sqrt{6}}(U_{+1}V_{-1} + 2U_0V_0 + U_{-1}V_{+1})$

Ex. $Y_2^0 = \sqrt{\frac{5}{16\pi}} \frac{3z^2 - r^2}{r^2}$.

Since $3z^2 - r^2 = 2z^2 + 2\left[-\frac{(x+iy)}{\sqrt{2}}\frac{(x-iy)}{\sqrt{2}}\right]$,

Y_2^0 is a special case of $T_0^{(2)}$ for $\vec{U} = \vec{V} = \vec{r}$.